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The run-off condition for coating and rimming flows

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A layer of liquid can be supported on the inside or outside of a horizontal rotating cylinder if the viscous forces pulling the liquid around with the cylinder are large enough to overcome the force of gravity. If there are places on the cylinder where the thickness of the layer is larger than a critical value, the excess fluid will run off. For a given maximum thickness the critical condition may be expressed as the minimum speed at which the given layer can be maintained. An approximation of the critical condition using lubrication theory was given by Wallis (1969) and by Deiber & Cerro (1976) for rimming flow and by Moffatt (1977) for coating and rimming flow. Here we address the question of the axial variations of the free surface on the coating layers, and show that they are dominated by the same type of balance between capillarity and centripetal acceleration which determines the shape of rotating drops and bubbles in the absence of gravity. The main results of this paper are the experiments which establish the validity of approximations used to describe the underlying fluid mechanics involved in rimming and coating flows.

1. Introduction

The aim of this paper is to describe in mathematical terms the steady flow of liquids which coat the outside or inside of rotating cylinders when gravity is not negligible. In the paper we draw together the work of Joseph & Preziosi (1987) on the axisymmetric interface shapes which arise under zero gravity when the liquid films rotate rigidly, and the work of Moffatt (1977), in which a lubrication analysis is used to express the effects of gravity in a two-dimensional flow in which axial variations of the interfaces are neglected.

One of the interesting achievements of the analysis of Moffatt is the derivation of a load condition approximating an answer to the following question. 'It is a matter of common experience if a knife is dipped in honey and then held horizontally the honey will drain off: but the honey may be retained on the knife by simply rotating it about its length. The question arises: What is the maximum load of honey that can be supported per unit length of knife for a given rotation rate?'

One of the main achievements of this paper is the verification of the maximum load condition from the lubrication analysis and the experimental comparisons, which establish that the approximations of the lubrication analysis are appropriate in a certain explicitly stated range of parameters. It is of some interest that our load condition is expressed as limiting value of the coating thickness. The films which coat cylinders are never two-dimensional and the axial variations which are observed are not weak. Hence the maximum thickness varies from point to point along the axis.

Run-off occurs locally, near points, in planes, where the maximum thickness has exceeded the critical thickness.

The aforementioned considerations oblige us to consider an extension of Moffatt's analysis in which the free surface $r = R(\theta, x)$ and the velocity $V = V(r, \theta, x)$ depend on the axial coordinate x. This we accomplish in a zeroth-order approximation in §4 by requiring that profile shape in a vertical plane where the effects of gravity are weakest should minimize the interfacial potential introduced by Joseph *et al.* (1985).

A two-dimensional analysis of rimming flow as a regular perturbation of rigid motion $\boldsymbol{V} = \boldsymbol{e}_\theta \Omega r$ with a small gravity parameter $g/\Omega^2 a$ has been given by Ruschak & Scriven (1976). They did a domain perturbation and solved the linear partial differential equations which arise at leading order in the concentric annular domain of the unperturbed problem. Choosing D as the thickness of the liquid-layer at zero gravity, they describe their results in terms of Reynolds number $R_\Omega = \Omega D^2/\nu$ and a dimensionless thickness $h = D/a$. In every case the maximum film thickness occurs in the upper quadrant $0 \leqslant \theta \leqslant \frac{1}{2}\pi$ on the rising side of the cylinder and the minimum thickness is diametrically opposite. For small values of R_Ω, the maximum thickness lies in a horizontal plane through the centre $\theta = 0$. The gravity-induced secondary motion vanishes at high rates of rotation and for thin films $h \rightarrow 0$.

A fully nonlinear finite element analysis of the two-dimensional rimming-flow problem was given by Orr & Scriven (1978). They found remarkable agreement between the solution of the linear and nonlinear problems even when the gravity parameter $g/\Omega^2 a$ was not small. At higher Reynolds number, however, the agreement is less good especially when the surface tension is also large.

An analysis of rimming flow of viscoelastic fluids was given by Sanders, Joseph & Beavers (1981). They did a domain perturbation like Ruschak & Scriven and showed how the position of the maximum thickness varies with the complex viscosity. They gave experimental results and showed how the storage and loss modulus could be backed out of the experimental results.

The essentials of the lubrication analysis used to derive the condition for run-off was given by Van Rossum (1958). He studied the case in which the liquid is lifted by means of a vertical plate emerging from the liquid. A lubrication analysis leading to run-off was presented as a homework problem (6.13 on p. 157) by Wallis (1969). It was also applied to rimming flow by Deiber & Cerro (1976).

All of the aforementioned authors restrict their attention to the two-dimensional problem in which axial variations are suppressed. Lubrication analysis of this problem yields the maximum load carried by a cylinder rotating at a given speed or the minimum speed at which a given load can be maintained. The critical load condition cannot be obtained by the method of domain perturbations used by Ruschak & Scriven (1976). Presumably the load condition could be simulated numerically, but there is no hint of the critical phenomena evident in the finite element results displayed by Orr & Scriven (1978). On the other hand, the lubrication analysis is less general than the domain perturbation study because it is restricted to a low-Reynolds-number approximation in which the position of the maximum and minimum thickness is confined to the horizontal plane through the axis of the cylinder.

Nearly all the aforementioned authors allude to three-dimensional effects in which there are large, even dramatic, variations of the interface shape along the axis of the cylinder. Large axial variations are also seen in coating and rimming flows of two liquids and one liquid and air when gravity is negligible. Photographs of large axial variations in coating flows can be found in this paper and in the papers by Yih (1960), Moffat (1977), Kovac & Balmer (1980) and Joseph & Preziosi (1987). Large axial variations in rimming flows are shown in photographs in this paper, in the papers of Balmer (1970), Karweit & Corrsin (1975) and in the paper of Sanders *et al.* (1981).

Joseph *et al.* (1985) and Joseph & Preziosi (1987) have studied the stability of two fluids between cylinders which rotate steadily with the same angular velocity. They find that rigid motion of the two fluids is always unconditionally stable and the interface between the two fluids minimizes an interfacial potential in which energies associated with surface tension and centripetal acceleration compete. The interfacial potential depends on a parameter

$$J = \frac{\Delta\rho\Omega^2\bar{d}^3}{T}, \tag{1.1}$$

where $\bar{d}$ is a mean radius with respect to volume, T is the surface tension and the density difference $\Delta\rho > 0$ when heavy fluid is outside (rimming flow) and $\Delta\rho < 0$ when heavy fluid is inside (coating flow). If $J > 4$, and only if $J > 4$, then the interface has a constant radius; if $J < 4$ there are large axial variations and the interface will always touch the cylinder either at a contact line, or with a tangent contact at the wetted rod, as in figure 4.

It follows that interfaces with $J < 4$ are never of constant radius even though both fluids rotate rigidly. When $J > 4$ the interface will have a constant radius provided that the bubble of light fluid on the inside is restrained from elongating by endwalls (see figure 4 of Joseph & Preziosi 1987).

We are going to show that the large axial variations which appear in coating and rimming flows when gravity is not negligible are essentially controlled by the aforementioned competition between surface tension and centripetal acceleration, with only weak second-order effects of gravity. The importance of this competition was mentioned in the penultimate paragraph of the paper by Moffatt (1977). Some aspects of the underlying stability problem can be found in Hynes (1979).

2. Controlling parameters for coating and rimming flow

To study coating and rimming flows when gravity is not negligible, we decompose the velocity $\boldsymbol{V}$ into a rigid motion $\boldsymbol{e}_\theta \Omega r$ plus a perturbation $\boldsymbol{u}$, which need not be small:

$$\boldsymbol{V} = \boldsymbol{e}_\theta \Omega r + \boldsymbol{u}. \tag{2.1}$$

The components of $\boldsymbol{u}$ are (u, v, w) in cylindrical coordinates r, θ, x and

$$\boldsymbol{u} = 0 \quad \text{on } r = a. \tag{2.2}$$

On the free surface $F = r - R(\theta, x) = 0$, the normal component of the velocity must vanish,

$$\boldsymbol{V}\cdot\boldsymbol{n} = \boldsymbol{e}_\theta\cdot\boldsymbol{n}\Omega r + \boldsymbol{u}\cdot\boldsymbol{n} = 0, \tag{2.3}$$

where $\boldsymbol{n} = \nabla F/|\nabla F|$, the shear stress must vanish and the normal stress is balanced by surface tension. Thus

$$-p\boldsymbol{n} + 2\mu\boldsymbol{D}[\boldsymbol{u}]\cdot\boldsymbol{n} = \rho\tfrac{1}{2}(\Omega^2 R^2)\,\boldsymbol{n} + 2HT\boldsymbol{n}, \tag{2.4}$$

where $\boldsymbol{D}[\boldsymbol{u}]$ is the symmetric part of $\nabla\boldsymbol{u}$, p is the pressure associated with $\boldsymbol{u}$, $2H$ is the sum of the principal curvatures on $r = R$ and T is the surface tension coefficient.

The approximations which we will implement to study coating and rimming flow decompose into an interfacial problem for axisymmetric shapes generated by rigid motion with $\boldsymbol{u} = 0$ and an azimuthal lubrication problem with $\boldsymbol{u} = \boldsymbol{e}_\theta v(r, \theta, x)$ where derivatives with respect to θ and x are assumed to be small relative to r derivatives and $\boldsymbol{n}\cdot\boldsymbol{e}_\theta \ll 1$. The equations (2.2), (2.3) and (2.4) and the equations of motion are satisfied identically with $\boldsymbol{u} = 0$ by the axisymmetric solutions given by Joseph & Preziosi (1987). For $\boldsymbol{u} = \boldsymbol{e}_\theta v(r, \theta, x)$ we satisfy (2.2) identically by requiring that

$$v(a, \theta, x) = 0, \tag{2.5}$$

(2.3) is satisfied approximately when $\boldsymbol{n}\cdot\boldsymbol{e}_\theta \ll 1$ and this is enough also to guarantee that $\boldsymbol{n}\cdot\boldsymbol{D}[\boldsymbol{e}_\theta v]\cdot\boldsymbol{n} \ll 1$. The tangential component of (2.4) will then vanish approximately if

$$D_{r\theta} = \tfrac{1}{2} r \frac{\partial v/r}{\partial r} = 0, \tag{2.6}$$

on $r = R$.

The equation of motion for v, without approximation, is

$$u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \theta} + w\frac{\partial v}{\partial z} + \frac{uv}{r} + \Omega\left\{2u + \frac{\partial v}{\partial \theta}\right\}$$
$$= -g\cos\theta - \frac{1}{\rho r}\frac{\partial p}{\partial \theta} + \nu\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) + \frac{\partial^2 v}{\partial x^2} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial u}{\partial \theta} - \frac{v}{r^2}\right\}. \tag{2.7}$$

The deviation v from rigid motion is forced by gravity. If the maximum thickness $D_0 = |R_0 - a|$ of the liquid coating the rotating cylinder is small relative to the cylinder radius a, we may neglect secondary motions and θ and x derivatives. This leads to

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2} = \frac{g}{\nu}\cos\theta, \tag{2.8}$$

which is to be solved relative to (2.5) and (2.6). The solution of this problem can be obtained as a perturbation in powers of $h_0 = D_0/a$ of the solution of the lubrication problem

$$\left.\begin{aligned} \frac{\partial^2 v}{\partial r^2} &= \frac{g}{\nu}\cos\theta, \\ v &= 0 \quad \text{at } r = 0, \\ \frac{\partial v}{\partial r} &= 0 \quad \text{at } r = R. \end{aligned}\right\} \tag{2.9}$$

Equations (2.9) show that at leading order v scales with gD_0^2/ν. Noting next that $\partial v/\partial\theta$ scales with v we may use the continuity equation to show that when $h_0 = D_0/a$ is small we may scale u and w with $h_0 g D_0^2/\nu$.

To find the domain of parameters in which (2.9) may be an approximation of the true v, change the variables in (2.7) as follows:

$$r = \begin{cases} a + D_0\bar{y} & \text{coating flow,} \\ a - D_0\bar{y} & \text{rimming flow,} \end{cases} \tag{2.10}$$
$$z = a\bar{z},$$

and

$$(v, u, w) = \frac{gD_0^2}{\nu}(\bar{v}, h_0\bar{u}, h_0\bar{w}). \tag{2.11}$$

The equation governing $\bar{v}$ is obtained by inserting (2.10) and (2.11) into (2.7). Thus

$$R_g\left\{\pm\bar{u}\frac{\partial\bar{v}}{\partial\bar{y}}+\frac{\bar{v}}{m}\frac{\partial\bar{v}}{\partial\theta}+\frac{h_0\bar{u}\bar{v}}{m}+h_0\,\bar{w}\frac{\partial\bar{v}}{\partial\bar{x}}\right\}$$

$$+R_\Omega\left\{\frac{\partial\bar{v}}{\partial\theta}+2h_0\,\bar{u}\right\}=-\cos\theta-\frac{1}{m}\frac{\partial\bar{p}}{\partial\theta}$$

$$+\frac{\partial^2\bar{v}}{\partial\bar{y}^2}\pm\frac{h_0}{m}\frac{\partial\bar{v}}{\partial\bar{y}}+h_0^2\frac{\partial^2\bar{v}}{\partial\bar{x}^2}+\frac{h_0^2}{m^2}\frac{\partial^2\bar{v}}{\partial\theta^2}$$

$$+\frac{2h_0^3}{m^2}\frac{\partial\bar{u}}{\partial\theta}-\frac{h_0^2}{m}\bar{v} \tag{2.12}$$

where

$$\left.\begin{aligned} m=1\pm h_0\,\bar{y},\quad h_0=\frac{D_0}{a},\\ R_g=\frac{gD_0^4}{a\nu^2},\quad R_\Omega=\frac{\Omega D_0^2}{\nu},\end{aligned}\right\} \tag{2.13}$$

and the upper sign is for coating flow. The Reynolds number R_g measures the importance of nonlinear terms while R_Ω multiplies linear terms which were retained in the perturbation analysis of Ruschak & Scriven (1976). The effect of increasing R_Ω is to shift the position of the maximum thickness into the first quadrant.

Boundary conditions for $\bar{v}$ are

$$\bar{v}=0\quad\text{at }\bar{y}=0,$$

and

$$\frac{\partial\bar{v}}{\partial\bar{y}}-h_0\frac{\bar{v}}{m}=0\quad\text{at }\bar{y}=\frac{D}{D_0}.$$

If we let $h_0\to 0$ and put the two Reynolds numbers R_g and R_Ω to zero we get

$$\left.\begin{aligned} \frac{\partial^2\bar{v}}{\partial\bar{y}^2}=\cos\theta,\quad 0\leqslant\bar{y}\leqslant\frac{D}{D_0},\\ \bar{v}(0)=0,\\ \frac{\partial\bar{v}}{\partial\bar{y}}=0\quad\text{at }\bar{y}=\frac{D}{D_0}.\end{aligned}\right\} \tag{2.14}$$

Equations (2.14) are rescaled versions of (2.9).

The parameter

$$J=\frac{\rho\Omega^2\bar{d}^3}{T},$$

introduced at (1.1) can be regarded as the ratio of pressure forces induced by the angular momentum of the rigid rotation to the hoop stress $T/\bar{d}$ associated with surface tension. This parameter does not enter in (2.14), but it controls the variation of the shape of the free surface with x.

We have thus identified four independent parameters for rimming and coating flows

$$(h_0,R_g,R_\Omega,J).$$

3. The run-off condition

The $v(r)$ satisfying (2.9) is in the form

$$v = \frac{g}{\nu}[\tfrac{1}{2}y^2 - Dy]\cos\theta, \tag{3.1}$$

where $y = r-a, \quad D = R-a$ (coating flow)

or $y = a-r, \quad D = a-R$ (rimming flow).

The total azimuthal velocity is $V = \Omega r + v$ or

$$V = \Omega a \pm \Omega y + v. \tag{3.2}$$

The term Ωy is small relative to Ωa since $0 \leqslant y \leqslant D$ and $h = D/a$ is small. Then, to within terms of order h, we have

$$V = \Omega a + \frac{g}{\nu}[\tfrac{1}{2}y^2 - Dy]\cos\theta. \tag{3.3}$$

The surface velocity

$$V = \Omega a - \frac{g}{\nu}\tfrac{1}{2}D^2\cos\theta, \tag{3.4}$$

is minimum at $\theta = 0$ where $D = D_0$.

The shape of the free surface follows from the statement that the volume flux Q is a constant where

$$Q = \int_0^D V\,\mathrm{d}y = \Omega a D - \frac{g}{\nu}\tfrac{1}{3}D^3\cos\theta = \Omega a D_0 - \frac{g}{\nu}\tfrac{1}{3}D_0^3. \tag{3.5}$$

This equation determines allowed values of

$$D(\theta) > 0, \quad D(0) = D_0, \quad 0 \leqslant \theta \leqslant \pi,$$

provided it can be solved. To apply the implicit function theorem we define the function

$$H(D, D_0, \theta) = \Omega a(D - D_0) - \frac{g}{3\nu}(D^3\cos\theta - D_0^3). \tag{3.6}$$

The equation $H = 0$ may be solved for D, uniformly in θ, provided that

$$\frac{\partial H}{\partial D} = \Omega a - \frac{gD^2}{\nu}\cos\theta > 0, \tag{3.7}$$

that is, whenever

$$h_0^2 S = \frac{D_0^2 g}{\nu\Omega a} < 1, \tag{3.8}$$

where $h_0 = D_0/a$. It is well known (Van Rossum 1958, figure 10) and easy to show that there are two positive solutions of (3.6) when (3.8) holds; the one with the smaller D agrees with experiments. The criterion (3.8), expressed differently, is given by (16) in Moffatt (1977).

When the equality $h_0^2 S = 1$ is satisfied, $\partial H/\partial D = 0$ at $D = D_0$. There is no steady single-valued solution of (2.9) with

$$h_0^2 S = \frac{R_g}{R_\Omega} > 1. \tag{3.9}$$

The criterion (3.9) is the same for coating and rimming flows. This criterion cannot be expected to hold when the conditions for the validity of lubrication theory are

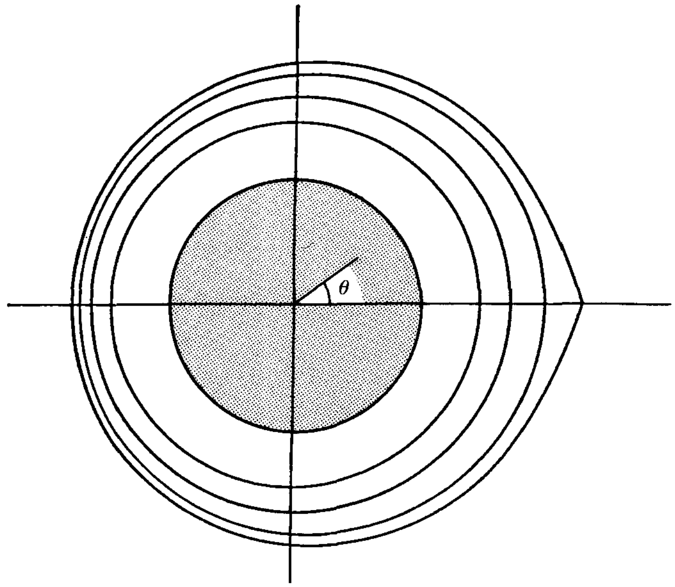


FIGURE 1. Interface between liquid and air in coating flow as given by (3.10) for $S = 0.6$ and different values of $h_0 = 0.5$, 0.75, 1.0 and 1.29. In this figure gravity points downward and the maximum and minimum thickness is at $\theta = 0$ and π, respectively, where θ is measured counterclockwise from the horizontal. We draw attention to the fact that thin films (say, $h_0 < 0.5$) are essentially axisymmetric.

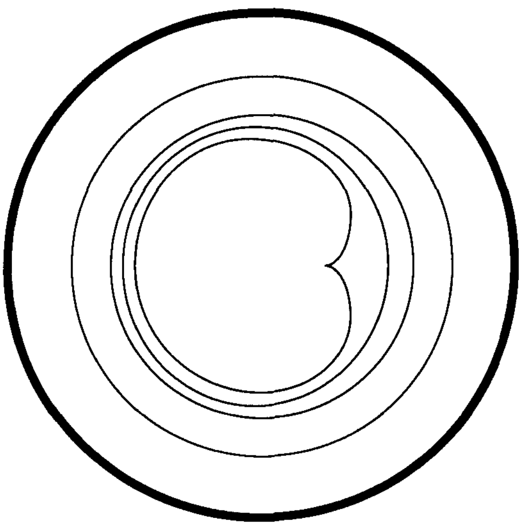


FIGURE 2. Interface between liquid and air in rimming flow as given by (3.10) for $S = 2$ and different values of $h_0 = 0.2$, 0.4, 0.55 and 0.707. Gravity is vertical and $\theta > 0$ is counterclockwise from the horizontal, as in figure 1.

violated, say, for large values of $h_0 = D_0/a$. The lubrication theory seems to work well beyond its expected range of validity, see figures 3, 4 and 7. In the case of rimming flow the criterion (3.9) makes no sense when the gap D_0 is larger than the radius a of the cylinder. Hence we cannot find a critical run-off condition, correct or incorrect, for values of $S < 1$.

Fluid	Silicone oils				Soybean oil	Amoco oil SAE 30	Amico gear lubricant 140	STP
μ (P)	0.2	10.3	125	1000	0.46	2.83	16.3	143
ρ (g/cm^3)	0.949	0.967	0.975	0.977	0.922	0.924	0.92	0.858
$T\left(\frac{\text{dynes}}{\text{cm}}\right)$	20.6	21.2	21.5	21.5	27	27	28	30

TABLE 1. Material parameters of test liquid (viscosity μ, density ρ, surface tension T)

In figure 1 we have plotted the smaller of the two solutions of (3.6) $h(\theta) = D(\theta)/a$ in coating flows for $S = 0.6$ and different values of $h_0 = D_0/a$ using (3.5) in dimensionless variables

$$h - S\tfrac{1}{3}h^3 \cos\theta = h_0 - S\tfrac{1}{3}h_0^3. \tag{3.10}$$

The graph of $h(\theta)$ for the limiting value $h_0 = (1/S)^{\frac{1}{2}} = \frac{5^{\frac{1}{2}}}{3}$ has a corner. In figure 2 we have plotted $h(\theta)$ for rimming flows for $S = 2$ and different values of h_0.

At the next order, the critical condition may be expressed as

$$h_0^2 S = 1 \mp \tfrac{1}{2}h_0 + O(h_0^2),$$

where the minus sign is for rimming flow (Preziosi 1986).

4. Coating flows

We did some experiments with oil layers coated on different rods rotating in air. The material parameters of the test liquids are listed in table 1. The experiments are like those described in §5 of the paper by Joseph & Preziosi (1987). The apparatus used in our experiments was nearly identical to the one used in the experiments of Moffatt (1977). A layer of liquid was first coated on the cylinder by rotating it while partially immersed in a trough; the roller was then raised from the trough while still rotating. The coating films achieved in this way could be maintained indefinitely. Once the rod is coated with oil, it stays coated; dry patches do not develop. Uniform coats are unstable. In the absence of important effects of gravity, undulating axisymmetric figures which touch the wetted cylinder with a tangent contact are observed (see figure 4). We are going to show that the major effects of gravity are confined to motions in the azimuth, the axial variations continue to be controlled essentially by the balance of forces competing in the minimization of the interfacial potential.

4.1. *Experimental verification of the run-off condition for coating flow*

We first establish a steady state with coated rods in which secondary motions driven by gravity need not be negligible. The maximum thickness of the coat in azimuthal planes varies more or less periodically along the cylinder as in figure 4. We then reduce the speed to the critical value where the liquid falls off the coat at the points of maximum thickness. The speed Ω and the maximum thickness ratio $h_0 = D_0/a$ is recorded at the critical condition at which run-off points of maximum bulge are observed. The comparison is exhibited in figure 3. The solid line is a graph of the

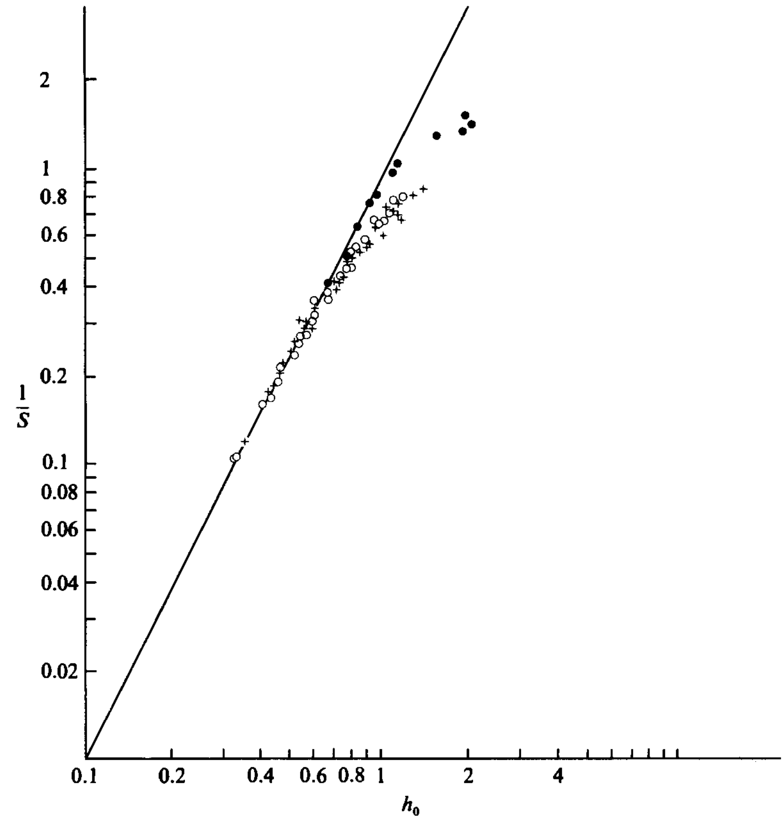


FIGURE 3. Comparison of the critical criterion (3.9) for run-off in coating flow with experiments. The solid line $Sh_0^2 = 1$ is theoretical: ●, silicone 1000 P; ○, STP 166 P; +, silicone 125 P.

theoretical critical condition $Sh_0^2 = R_g/R_\Omega = 1$. The experiments are points which lie near this line or in the region $R_g > R_\Omega$ where nonlinear effects have become noticeable. The largest value of $R_g = R_\Omega$ on the $Sh_0^2 = 1$ is 2.6×10^{-3}. Some values of R_g at experimental points furthest in the nonlinear region $R_g > R_\Omega$ are 0.034, 0.014 and 0.0019 for silicone oil 12500, STP and silicone 100000 respectively. We already noted, under (2.13), that the linear theory of Ruschak & Scriven predicts the rotation of the point of maximum thickness into the first quadrant with increasing R_Ω. We observed small changes, less than ten degrees, in this direction.

4.2. *Shape computation based on the decomposition into rigid motion plus a lubrication approximation*

In figure 4 we exhibit results which show that the decomposition $\boldsymbol{v} = \boldsymbol{e}_\theta(\Omega r + v)$, with v determined by lubrication theory, gives rise to theoretical predictions of shapes which agree with experiments even when gravity is not negligible. The theoretical prediction is given by the dots. It is necessary to explain how the theory is computed. We first assume that formula (3.10), giving the shape of the free surface in a plane perpendicular to the axis of the rod, is valid at each x along the rod, with $h(\pm\frac{1}{2}\pi, x)$ determined by minimizing the interfacial potential for rigid motions with negligible gravity, using the theory of Joseph & Preziosi (1987). To use this latter theory it is necessary to give the value of the parameter $J = \rho\Omega^2\bar{d}^3/T$ where $\bar{d}$ is the mean radius

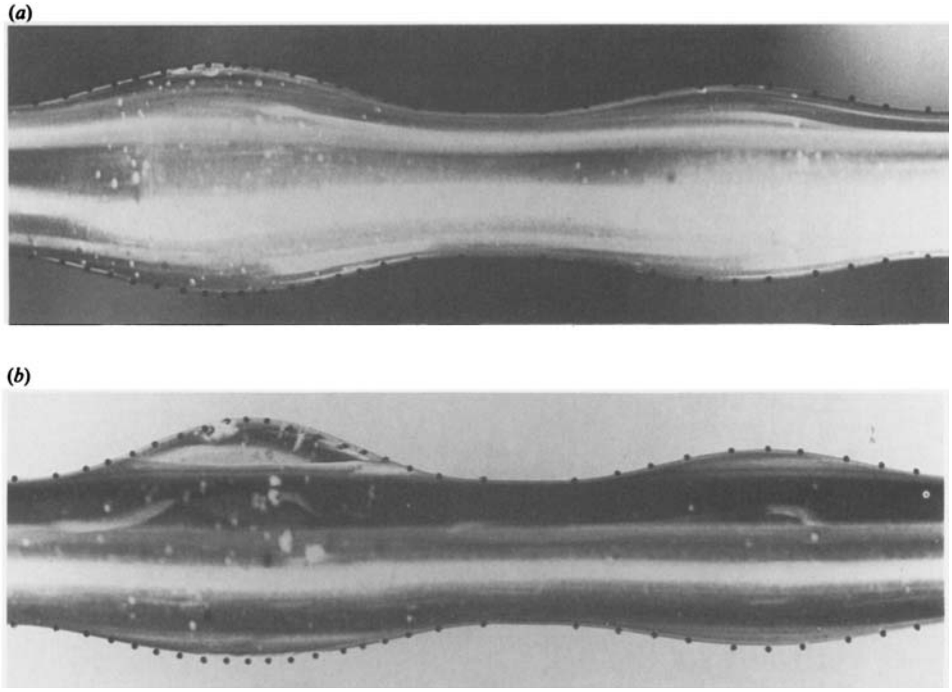


FIGURE 4. (a) Coating flow of 1000 p silicone oil on a rotating rod, $a = 0.716$ cm, $\Omega = 3.35$ rad/s; front view, $\theta = \pm\frac{1}{2}\pi$. There is very little bulging evident. The cross-section is as if the entire figure were axisymmetric. The dots are computed by minimizing the interfacial potential as if the figure were axisymmetric. (b) Coating flow of 1000 p silicone oil on a rotating rod, $a = 0.716$ cm, $\Omega = 3.35$ rad/s. Top view. The out-of-roundness associated with secondary motion due to gravity is evident. The fluid accumulates at $\theta = 0$ where the action of gravity opposes rotation. The flow is perfectly steady in laboratory coordinates. The dots are computed from lubrication theory and are not in perfect agreement.

with respect to volume. Since the density ρ and surface tension T of the liquid are known and Ω is given, we need to determine $\bar{d}$. We do this by identifying the volume of one drop which is the volume of liquid inside two rings where the free surface makes a tangent contact with the wetted rod, as in figure 4. The two drops shown in figure 4 have different $\bar{d}$ and the computation must be carried out for each drop separately. With $\bar{d}$ given, we compute an axisymmetric figure $r = R(x)$ by minimizing the interfacial potential subject to the constraint that $R'(a) = 0$. This gives the profiles shown as dots in figure 4(a) (front view). Now we write $r = R(\theta, x)$ for the shape and identify the shape in figure 4(a) as the one in the plane $\theta = \pm\frac{1}{2}\pi$, which is the cross-section in the plane of gravity and axis of rotation. This gives a function, $R(\frac{1}{2}\pi, x)$ or $h(\frac{1}{2}\pi, x)$ which, when substituted in (3.10)

$$h(\tfrac{1}{2}\pi, x) = h_0 - S\tfrac{1}{3}h_0^3,$$

determines h_0. The shape formula then is

$$h(\theta, x) - \tfrac{1}{3}Sh^3(\theta, x)\cos\theta = h(\tfrac{1}{2}\pi, x)$$

and the dots in figure 4(b) are computed from this formula.

5. Rimming flow

Rimming flow is a coating flow inside a rotating cylinder. The apparatus used in our experiments on rimming flows was a cylindrical Plexiglas container with inner radius 1.27 cm, closed at the ends, with an end-to-end distance of 15.24 cm.

5.1. *Axial structures, bubbles and disking*

If Ω is large, the effects of gravity are relatively small, leaving a uniform layer of radius D rimming the cylinder, rotating with it as a rigid body. Gravity can be considered a small perturbation of rigid rotation when $Sh_0^2 = gD_0^2/a\nu\Omega$ is small (thin films of viscous liquids) or when the effects of 'centrifugal gravity' overwhelm terrestrial gravity $\Omega^2 D_0/2g \gg 1$. The shape of free surfaces in such perturbed problems is naturally close to shapes which arise in the unperturbed motion. In rigid motions a uniform layer can occur only if $J = \rho\Omega^2 D^3/T > 4$. If $J < 4$, rigid motion must take form as rotating bubbles of air in liquid, perturbed by gravity. The length of the bubbles decreases with J. In figure 5 we have exhibited photographs of air bubbles in STP for (*a*) $J = 3.1$, (*b*) $J = 2.0$ and (*c*) $J = 0.8$ where $\bar{D}$ is the mean bubble radius. We compute $\bar{D} = D$ when Ω is so large that the STP is centrifuged into a uniform layer inside the rotating cylinder. These STP bubbles appear to be axisymmetric to the naked eye (S is small because $\nu = 166$ P is large) so that large effects of gravity are not noticeable. In figure 6 we have exhibited top and side views of air bubbles in soybean oil (brand name, Crisco). The effects of gravity are not negligible. The bubbles are displaced towards the side of the cylinder where the motion due to gravity and rotation add. The slower moving liquid on the other side where gravity opposes rotation leads to a much larger accumulation of liquid, as in the coating flow shown in figure 4(*b*). When the angular velocity of the fluid is small, most of the liquid lies in a pool at the bottom of the cylinder, and the coating layer is very thin. The minimum thickness of the layer increases with Ω and the pool at the bottom is rotated a little towards the ascending side of the cylinder. If $J < 4$ and the minimum thickness is large enough, axial structures will develop at the value of θ on the descending side of the cylinder where the thickness is minimum (see figure 6*a*) and if there is enough fluid there, bubbles will form. In this case we may see bubbles under conditions in which all the fluid in the pool has not run up onto the cylinder. At higher values of Ω the run-up condition will be satisfied and the motion will be much closer to rigid rotation. There is evidently then an effective J such that there will be bubbles if $J < 4$ and no bubbles if $J > 4$ (see figure 7).

The form of the bubbles of air in soybean oil shown in figure 6 resembles the photograph of disking shown in figure 2 of Karweit & Corrsin (1975). They use the word disking to refer to the disk-like structures which separate bubbles of air. They observed these disks in cylinders from 5% to nearly full of liquid. They do not specify the fluids used, nor do they give any data on angular velocity or layer thickness. The explanation of the formation of these disks or bubbles is based on liquid falling from overhead and 'draping' complete liquid curtains across the cross-section of the cylinder. First they observed periodic axial structures on the bottom of the cylinder. They said that the period of the ribs depended on Ω, a, ρ, T and volume. Apart from a these are the parameters of J. 'The wavelength of the disking, on the other hand, is at most weakly dependent on Ω, and perhaps ρ and T. That these two phenomena be hydrodynamically connected is uncertain, although a relation is suggested by the fact that their lengthscales are nearly equal.' This last remark about equal

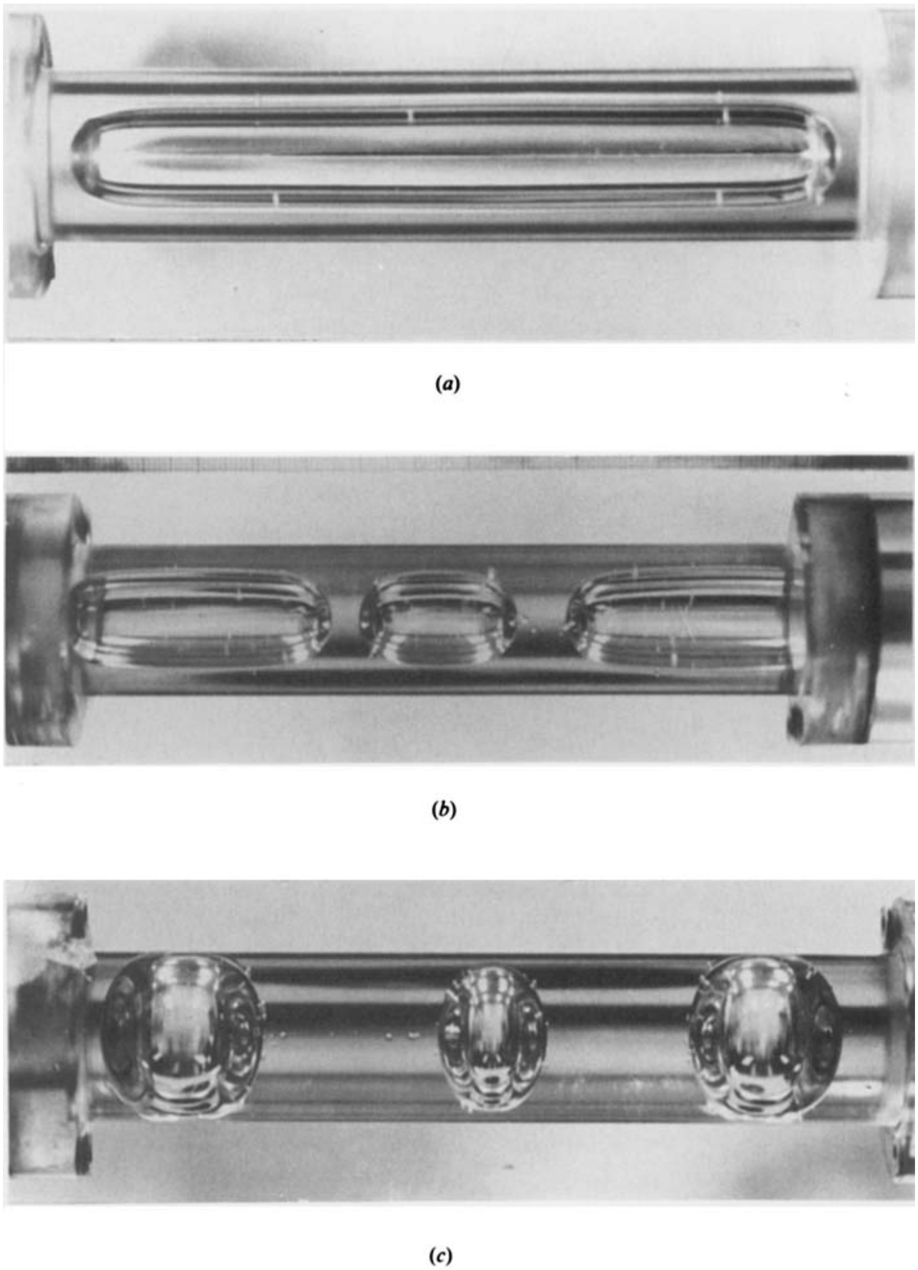
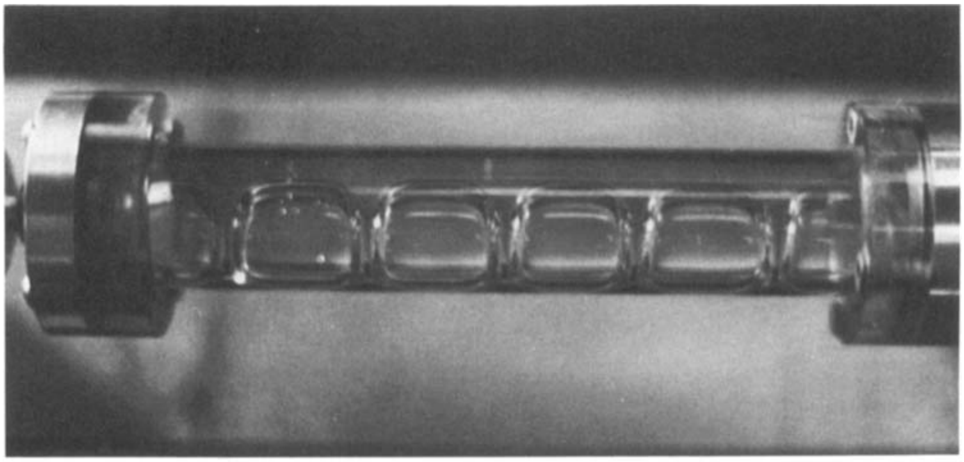


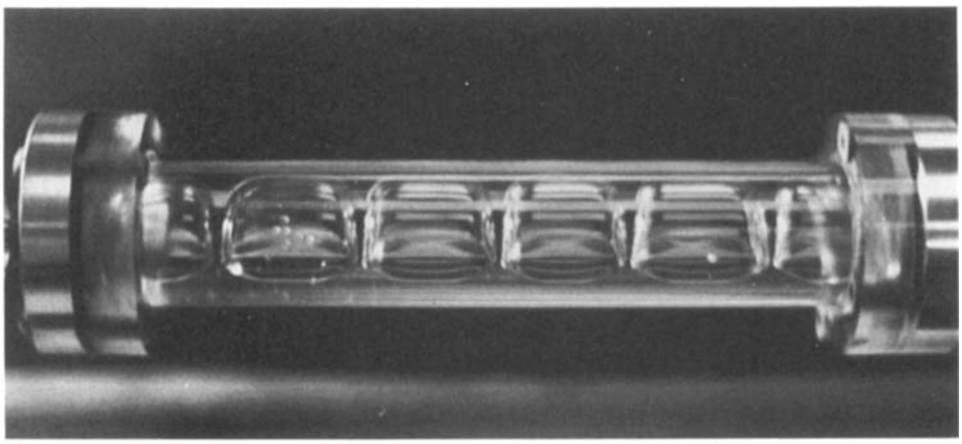
FIGURE 5. (a) Long bubble of air in STP: $\bar{h} = 0.55$, $\Omega = 28.3$ rad/s, $J = 3.1$. (b) The long bubble breaks into three as Ω is reduced: $\bar{h} = 0.55$, $\Omega = 20.9$ rad/s, $J = 2.0$. (c) The bubbles tend to assume a nearly spherical shape as the speed is reduced further: $\bar{h} = 0.55$, $\Omega = 14.1$ rad/s, $J = 0.8$.

lengthscales suggests that disking is actually a description of the formation of air bubbles associated in some way with minimizing an interfacial potential, with some effective J, governing the competition between surface tension and inertia.

'Disking' in STP was also observed by Sanders *et al.* (1981). It is probable that the motions in these experiments are much closer to rigid ones than in the experiments



(*a*)



(*b*)

FIGURE 6. Air bubbles in soybean oil. The layer of oil is thicker on the side of the cylinder rising against gravity. $\Omega = 600$ r.p.m., $\bar{h}_0 = 0.38$. (*a*) Top view. (*b*) Front view.

of Karweit & Corrsin because STP is very viscous. Sanders *et al.* show disking for two values of $\Omega = 0.17$ rev/s in their figure 11 and for $\Omega = 0.33$ rev/s in figure 12. It is hard to estimate the mean bubble radius $\bar{D}$ from their photographs but it is not greatly in error to take $\bar{D} = a$, the cylinder inner radius, from which one may compute $J = 1.09$ for the short cells shown in their figure 11(*f*) and $J = 4.01$ for the long cells shown in their figures 12(*e*) and (*f*). This data is consistent with the thought that the disking actually describes the formation of bubbles of air in the competition between interfacial tension and centripetal acceleration.

5.2. *Run-off and run-on*

Since any fluid which runs off the inside wall of a rotating cylinder must collect at the bottom of the cylinder it is convenient to do the run-off experiment in reverse.

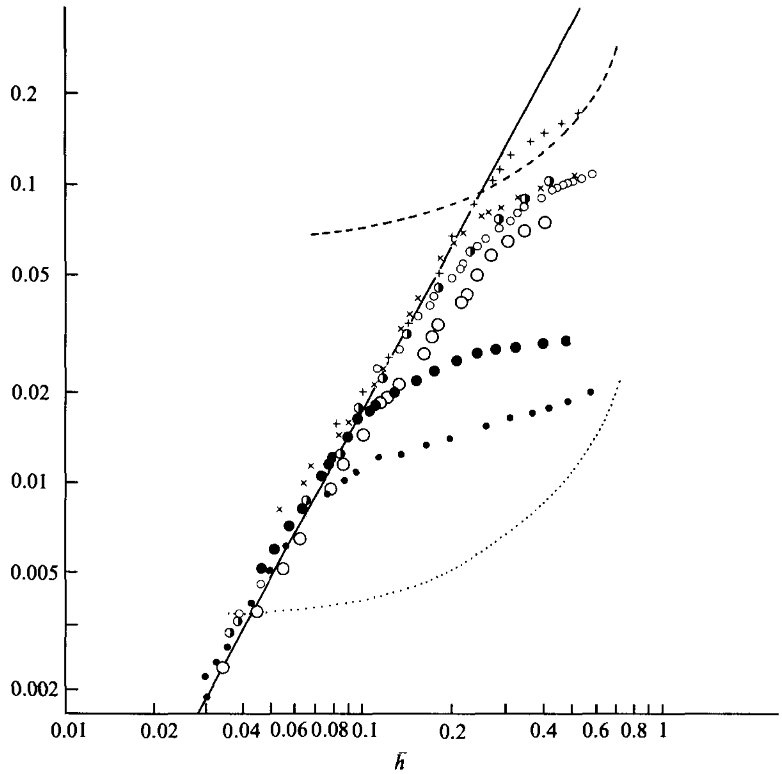


FIGURE 7. Comparison of the critical criterion (3.9) for run-off in rimming flow with experiments. The broken lines represent the line $J = 4$ for Soybean oil (···) and Silicone 1000 (---). The solid line $S\bar{h}_0^2 = 1$ is theoretical: +. STP; ◑, Amoco oil, SAE 30; ×, Silicone oil, 12500 cp; ○, Amoco gear lubricant 140; ○, Silicone oil, 1000 cp; •, Soybean oil; ●, Silicone oil 20 cp.

We get all the liquid which is at the bottom when Ω is small to run on the wall when Ω is increased to the critical value. There is one and the same critical value for run-off and run-on. The sequence of events in a run-on experiment may be described as follows. The cylinder is partially filled with liquid, usually a small amount. When Ω is small most of the liquid is at the bottom, but a thin layer coats the rotating wall. More fluid is captured by this layer as Ω is increased but there is a pool below, well-defined by an abrupt variation in the thickness of the layer, resembling a hydraulic jump. The pool of liquid is not drawn on the wall in a continuous manner. Instead, there is a critical condition, here passed at a critical Ω, for run-on. Our observations of run-on differ slightly from the observations of rimming by Karweit & Corrsin (1975) who do not mention a critical run-on condition. The events leading to run-on could not be described by a lubrication theory, but in the post-critical regime all the fluid is in a thin layer, of variable thickness, which is nearly in rigid motion. In this situation lubrication theory could apply for all conditions in which S is less than the critical value for run-off.

The experimental results are displayed in figure 7. The experiments are shown as points at which run-on occurs. We find a critical Ω and $\bar{h} = \bar{D}/a$ is a true-layer thickness in the high-speed case in which all the liquid is centrifuged onto a layer of uniform thickness. To compare with theory we need $h_0 = D_0/a$. We take the average of the $h(h_0, \theta)$ satisfying (3.10)

$$\bar{h}(h_0) = \frac{1}{2\pi}\int_0^{2\pi} h(h_0, \theta)\,\mathrm{d}\theta,$$

and equate it with $\bar{h}$ from experiments. We may also compute the area between $r = a$ and $r = R(\theta)$ from (3.10) and equate with areas of the annular ring of thickness $\bar{h}$. These two averages give nearly identical values of h_0. The maximum values of h_0, over x, are used in the experiment. The agreement is such as to leave no doubt that the lubrication approximations are essentially correct in a domain of parameters in which h_0 is not too large (or perhaps, too small) and in which the nonlinear terms measured by Reynolds number R_g are sufficiently small.

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